A Cauchy problem for a generalized wave equation

By H. BERENS and U. WESTPHAL in Aachen (B. R. Deutschland)[1])

Introduction

The equation

$$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \qquad (x, t > 0)$$

with the initial conditions

$$w(x,0) = \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0 \qquad (x > 0)$$

and the boundary conditions

$$w(0,t) = f(t), \quad \lim_{x\to\infty} w(x,t) = 0 \qquad (t > 0)$$

is no doubt the simplest type of a boundary value problem which may be formulated for the wave equation. Its solution is given by $w(x, t) = f(t-x)$ for $0 < x < t$ and $= 0$ for $x > t$, in case $f(t)$ is twice continuously differentiable. The object here is to study a problem of this type for a generalized wave equation in the Lebesgue space $L^p(0, \infty)$ $(1 \leqq p < \infty)$. For the sake of precision we first restate the original problem as follows:

Let the operator J^{-2}, with domain $D(J^{-2}; p) = \{f \in L^p(0, \infty);\ f$ and f' locally absolutely continuous on $[0, \infty)$ with $f(0) = f'(0) = 0$ and $f'' \in L^p(0, \infty)\}$ and range in $L^p(0, \infty)$ $(1 \leqq p < \infty)$, be defined by $J^{-2} f = f''$.

Cauchy problem I. Given an element $f_0 \in L^p(0, \infty)$, find a vector-valued function $w(x) = w(x; f_0)$ on $[0, \infty)$ to $L^p(0, \infty)$ such that

(i) $w(x)$ is twice continuously differentiable in the L^p-norm on $(0, \infty)$;

(ii) $w(x) \in D(J^{-2}; p)$ for each $x > 0$ and $\frac{d^2}{dx^2} w(x) = J^{-2} w(x)$;

(iii) there is a constant $M = M_{f_0} > 0$ such that $\|w(x)\|_p \leqq M$ $(x > 0)$.

(iv) $\lim_{x\to 0+} \|w(x) - f_0\|_p = 0$.

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One easily proves that for any given $f_0 \in D(J^{-2}; p)$ there is a unique solution $w(x; f_0) = W(x) f_0$, where $W(x)$ is the semi-group of right translations on $L^p(0, \infty)$:

$$[W(x)f](t) = \begin{cases} 0, & 0 < t \leqq x, \\ f(t-x), & x < t < \infty. \end{cases} \tag{1}$$

The Cauchy problem I will be generalized in such a way that the operator $J^{-2} = (\partial/\partial t)^2$ is replaced by a differential operator of order 2γ $(0 < \gamma \leqq 1)$, namely by $J^{-2\gamma} = (\partial/\partial t)^{2\gamma}$. (For the exact definition of $J^{-\gamma}$ and its domain in $L^p(0, \infty)$ see the following section.) In case $\gamma = 1/2$, this leads to a boundary value problem of the heat-conduction equation for a semi-infinite rod.

Abstract Cauchy problems of higher orders (especially of order two) are studied in E. Hille and R. S. Phillips [9, Sec. 23. 9]. Also we refer to Yu. I. Lyubich [11]. In particular for the wave equation we mention two papers [7, 8] of E. Hille. However, the cited papers always deal with initial value problems, while the above Cauchy problem I is a proper boundary value problem (cf. conditions (iii) and (iv)).

The Cauchy problem for the generalized wave equation will be treated in Sec. 2. To this end we first solve a corresponding Cauchy problem of order one via the Hille—Yosida theorem of semi-group theory. Thus we collect in Sec. 1 some results on semi-groups of operators on Banach spaces and then define integral and differential operators of fractional order $\gamma > 0$ acting on functions defined on the positive real axis. One-to-one characterizations of these notions are given via Laplace transforms, which also play an important role in the proofs. In Sec. 3 we finally prove several equivalent characterizations of the domain of the differential operator $J^{-\gamma}$ $(\gamma > 0)$ in the function space $L^p(0, \infty)$ $(1 \leqq p < \infty)$. The results obtained are generalizations of those presented by the authors in [4]. The most interesting characterization here will be in terms of the vector-valued integral

$$\int^{\infty} \frac{\Delta_u^n f}{u^{1+\gamma}}\, du \qquad (0 < \gamma < n), \tag{2}$$

where

$$[\Delta_u^n f](\cdot) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(\cdot - ju) \tag{3}$$

is the n-th Riemann difference for the semi-group of right translations (1) on $L^p(0, \infty)$.

The results established in Sec. 3 possess far-reaching generalizations to fractional powers of infinitesimal generators of semi-groups of operators on Banach spaces. For further details, we refer to Sec. 5 in [4], to [3] and the literature cited there.

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1. Auxiliary results

Let X be a real or complex Banach space with elements $f, g, \ldots$ and norm $\|\cdot\|$, and let $\mathfrak{E}(X)$ be the Banach algebra of endomorphisms of X. If $T \in \mathfrak{E}(X)$, $\|T\|$ denotes the norm of T. A family of operators $\{T(x);\, x \geqq 0\}$ in $\mathfrak{E}(X)$ is said to be a contraction semi-group of class $(\mathfrak{C}_0)$, if it is subject to the following conditions: (i) $T(0) = I$ (identity operator); (ii) $T(x_1 + x_2) = T(x_1)T(x_2)$ for $x_1, x_2 \in [0, \infty)$; (iii) $\|T(x)\| \leqq 1$ uniformly with respect to $x \geqq 0$; (iv) $\lim_{x \to 0+} \|T(x)f - f\| = 0$ for all $f \in X$. The infinitesimal generator A of $\{T(x);\, x \geqq 0\}$, defined by $Af = \operatorname{s-lim}_{x \to 0+} x^{-1} \cdot [T(x) - I]f$ whenever the strong limit (s-lim) exists, is a closed linear operator with domain $D(A)$ dense in X. The powers A^r of A ($r = 2, 3, \ldots$), are defined inductively. If f belongs to $D(A^r)$, so does $T(x)f$ for each $x \geqq 0$ and

$$\frac{d^r}{dx^r} T(x)f = A^r T(x)f = T(x)A^r f.$$

If $\{(Tx);\, x \geqq 0\}$ has a holomorphic extension $T(z)$ $(z = x + iy)$ in a sector $\{z;\, 0 < x < \infty, |\arg z| \leqq \alpha_0 < \pi/2\}$ of the complex plane, we speak of a holomorphic semi-group. A necessary and sufficient condition for this is that $T(x)[X] \subset D(A)$ for $x > 0$ and that there is a constant N such that $x\|AT(x)\| \leqq N$ for $x > 0$.

Under the above hypotheses upon $\{T(x);\, x \geqq 0\}$ the set $\{\lambda;\, \lambda > 0\}$ belongs to the resolvent set $\varrho(A)$ of the generator A, and the resolvent operator $R(\lambda; A)$ is given by

$$R(\lambda; A)f = \int_0^\infty e^{-\lambda x} T(x)f\, dx \qquad (f \in X;\, \lambda > 0). \tag{4}$$

Also the inversion formula

$$T(x)f = \operatorname*{s-lim}_{\lambda \to \infty} e^{-\lambda x} \sum_{j=0}^{\infty} \frac{(\lambda x)^j}{j!} [\lambda R(\lambda; A)]^j f \qquad (f \in X) \tag{5}$$

holds uniformly with respect to x in any finite interval $[0, b]$. Finally, let us formulate the Hille—Yosida theorem: A closed linear operator U with dense domain and range in a Banach space X generates a contraction semi-group $\{T(x);\, x \geqq 0\}$ of class $(\mathfrak{C}_0)$ in $\mathfrak{E}(X)$ if and only if $\{\lambda;\, \lambda > 0\} \subset \varrho(U)$ and $\lambda\|R(\lambda; U)\| \leqq 1$ for all $\lambda > 0$. Moreover, U is the infinitesimal generator of exactly one semi-group given by (5). For a treatment of semi-group theory see E. Hille and R. S. Phillips [9, Part II], K. Yosida [13, Ch. IX] or P. L. Butzer and H. Berens [5, Ch. I].

We now introduce the concepts of integration and differentiation of fractional order. In the following, $f(t)$ will always be a real or complex-valued Lebesgue-measurable function defined on the positive real axis. The integral of f of order

$\gamma>0$ is then defined by the (Laplace) convolution integral

(6) $$[J^\gamma f](t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} f(u)\, au \qquad (t>0).$$

$J^\gamma f$ exists for almost all t in $(0, \infty)$, whenever $f\in L(0, b)$ for every $b>0$, and also belongs to this space. On the other hand, for a $\gamma>0$ with $k-1<\gamma\leqq k$ (k integral) the derivative of f of order γ is defined by

(7) $$[J^{-\gamma} f](t) = \frac{d^k}{dt^k}[J^{k-\gamma} f](t) \qquad (t>0),$$

whenever this expression has a meaning, where $J^0=I$ (for the notations see the remarks in G. Doetsch [6, vol. III, p. 164]; moreover, we refer to the literature cited in [4]). The following lemma characterizes the fractional integral and derivative of a function f by means of the Laplace transform.

Lemma 1. *Let $\gamma>0$ and f, g be two functions in $L(0, b)$ for every $b>0$.*

(a) *If the Laplace integral of f, i.e.*

$$f^\wedge(s) = \mathfrak{L}[f](s) \equiv \int_0^\infty e^{-st} f(t)\, dt,$$

converges absolutely for every complex number s with $\operatorname{Re} s>0$, *so does* $\mathfrak{L}[J^\gamma f](s)$ *and*

$$[J^\gamma f]^\wedge(s) = s^{-\gamma} f^\wedge(s) \qquad (\operatorname{Re} s>0),$$

where the branch of s^γ is taken such that $\operatorname{Re} s^\gamma>0$, *when* $\operatorname{Re} s>0$.

(b) *If $f^\wedge(s)$ and $g^\wedge(s)$ exist in the absolute sense for* $\operatorname{Re} s>0$ *and if*

$$s^\gamma f^\wedge(s) = g^\wedge(s) \qquad (\operatorname{Re} s>0),$$

then for $k-1<\gamma\leqq k$ (k integral) $J^{k-\gamma}f$ and its derivatives $[J^{k-\gamma-1}f], \cdots, [J^{1-\gamma}f]$ up to the order $(k-1)$ are locally absolutely continuous on $[0, \infty)$ with $[J^{k-\gamma}f](0)=\cdots$ $\cdots=[J^{1-\gamma}f](0)=0$ and $[J^{-\gamma}f](t)=g(t)$ a.e.

For a proof see D. V. Widder [12, Ch. II, § 8].

Since we are mainly interested in functions f belonging to $L^p(0, \infty)$ $(1\leqq p<\infty)$ (endowed with the usual norm), we restrict the domain of $J^{-\gamma}$ $(\gamma>0)$ to the set

(8) $D(J^{-\gamma}; p)=\{f\in L^p(0, \infty);$ there is a $g\in L^p(0, \infty)$ with $f(t)=[J^\gamma g](t)$ a.e.$\}$.

Then $J^{-\gamma}f$ exists and is equal to g. By Lemma 1 we have equivalently

(9) $D(J^{-\gamma}; p)=\{f\in L^p(0, \infty);$ there is a $g\in L^p(0, \infty)$ with $s^\gamma f^\wedge(s)=g^\wedge(s)$ $(\operatorname{Re} s>0)\}$

and

(10) $$[J^{-\gamma}f]^\wedge(s) = s^\gamma f^\wedge(s) = g^\wedge(s) \qquad (\operatorname{Re} s>0).$$

2. The generalized wave equation

We now formulate and solve the Cauchy problem of first order for the operator $B_\gamma = -J^{-\gamma}$ $(0<\gamma<1)$ with domain $D(B_\gamma; p) = D(J^{-\gamma}; p)$ and range in $L^p(0, \infty)$ $(1 \leqq p < \infty)$.

Cauchy problem II. Given a function $f_0 \in L^p(0, \infty)$, find a function $w_\gamma(x) = w_\gamma(x; f_0)$ on $[0, \infty)$ to $L^p(0, \infty)$ such that

(i) $w_\gamma(x)$ is strongly continuously differentiable on $(0, \infty)$;

(ii) $w_\gamma(x) \in D(B_\gamma; p)$ and $(d/dx) w_\gamma(x) = B_\gamma w_\gamma(x)$ for each $x > 0$;

(iii) $\lim_{x\to 0+} \|w_\gamma(x) - f_0\|_p = 0$.

Proposition 2. (a) *B_γ is a closed linear operator with domain dense in $L^p(0, \infty)$.*

(b) *The set $\{\lambda; \lambda > 0\}$ belongs to the resolvent set $\varrho(B_\gamma)$ of B_γ, and the resolvent has the representation*

$$[R(\lambda; B_\gamma)f](t) = \int_0^t f(t-u) r_\gamma(\lambda; u)\, du \qquad (f \in L^p(0, \infty)), \tag{11}$$

where

$$r_\gamma(\lambda; t) = \frac{\sin \gamma\pi}{\pi} \int_0^\infty e^{-tu} \frac{u^\gamma}{\lambda^2 - 2\lambda u^\gamma \cos \gamma\pi + u^{2\gamma}}\, du. \tag{12}$$

Moreover,

$$\|R(\lambda; B_\gamma) f\|_p \leqq \frac{\|f\|_p}{\lambda} \qquad (f \in L^p(0, \infty)). \tag{13}$$

Proof. (a) The linearity of B_γ is obvious by definition. To prove that B_γ is closed, suppose there is a sequence $\{f_n\}_{n=1}^\infty$ in $D(B_\gamma; p)$ such that f_n and $B_\gamma f_n$ converge in the L^p-norm to an f_0 and g_0, respectively. Since strong convergence implies weak convergence, we have for each fixed s $(\mathrm{Re}\, s > 0)$

$$\lim_{n\to\infty} f_n^\wedge(s) = f_0^\wedge(s) \quad \text{and} \quad \lim_{n\to\infty} -s^\gamma f_n^\wedge(s) = -s^\gamma f_0^\wedge(s) = g_0^\wedge(s),$$

i.e. $f_0 \in D(B_\gamma; p)$ and $B_\gamma f_0 = g_0$. Finally, it is easy to see that $C_{00}^\infty(0, \infty)$, the space of arbitrarily often continuously differentiable functions with compact support in $(0, \infty)$, belongs to $D(B_\gamma; p)$. Since $C_{00}^\infty(0, \infty)$ is dense in $L^p(0, \infty)$, so is $D(B_\gamma; p)$.

(b) At first we prove that $\{\lambda; \lambda > 0\} \subset \varrho(B_\gamma)$, i.e. we have to show that for each $\lambda > 0$ the operator $\lambda I - B_\gamma$ from $D(B_\gamma; p)$ to $L^p(0, \infty)$ has an inverse $[\lambda I - B_\gamma]^{-1}$ such that its domain is equal to $L^p(0, \infty)$ (since B_γ is a closed operator). Indeed, the equation

$$[\lambda I - B_\gamma] f = \theta \quad \text{or, equivalently,} \quad \lambda f^\wedge(s) + s^\gamma f^\wedge(s) = 0 \quad (\mathrm{Re}\, s > 0) \tag{14}$$

implies that the function $f(t)$ is zero almost everywhere. Thus $[\lambda I - B_\gamma]^{-1}$ exists, and it remains to prove that for any given $g \in L^p(0, \infty)$ there is an $f \in D(B_\gamma; p)$ satisfying

(15) $\lambda f - B_\gamma f = g$ or, equivalently, $\lambda f^\wedge(s) + s^\gamma f^\wedge(s) = g^\wedge(s)$ (Re $s > 0$).

But the function $(\lambda + s^\gamma)^{-1}$ (Re $s > 0$) is the Laplace transform of the function $r_\gamma(\lambda; \cdot)$ defined in (12). $r_\gamma(\lambda; \cdot)$ is non-negative, belongs to $L^1(0, \infty)$ and $\int_0^\infty r_\gamma(\lambda; t)\,dt = \lambda^{-1}$ (see T. Kato, [10]). Hence the element

$$f(t) = [R_\lambda g](t) = \int_0^t g(t-u)\, r_\gamma(\lambda; u)\, du$$

belongs to $D(B_\gamma; p)$ and solves the differential equation (15). Thus $\{\lambda; \lambda > 0\} \subset \varrho(B_\gamma)$ and the resolvent $R(\lambda; B_\gamma)$ equals the operator R_λ. Finally,

$$\|R_\lambda g\|_p \leqq \|r_\gamma(\lambda; \cdot)\|_1 \|g\|_p = \lambda^{-1}\|g\|_p \qquad (\lambda > 0; \quad g \in L^p(0, \infty))$$

proving the estimate (13).

Proposition 2 shows that the operator B_γ with domain and range in $L^p(0, \infty)$, $(1 \leqq p < \infty)$ satisfies the assumptions of the Hille—Yosida theorem. This leads to

Theorem 3. *The Cauchy problem* II *has a unique solution* $w_\gamma(x; f) = W_\gamma(x) f$ $(x \geqq 0)$ *for any given* $f \in L^p(0, \infty)$. $\{W_\gamma(x); x \geqq 0\}$ *is a holomorphic contraction semi-group of class* $(\mathfrak{C}_0)$ *in* $\mathfrak{E}(L^p(0, \infty))$ *generated by* B_γ *and is given by the convolution integral*

$$[w_\gamma(x) f](t) = \int_0^t f(t-u)\, p_\gamma(x; u)\, du \qquad (f \in L^p(0, \infty)) \tag{16}$$

with kernel

$$p_\gamma(x; t) = \frac{1}{\pi} \int_0^\infty \exp(tu\cos\sigma - xu^\gamma \cos\gamma\sigma) \cdot \sin(tu\sin\sigma - xu^\gamma \sin\gamma\sigma + \sigma)\, du \tag{17}$$

$\left(x > 0,\ t \geqq 0;\ \frac{\pi}{2} \leqq \sigma \leqq \pi,\ 0 < \gamma < 1\right)$, *a Lévy stable density function on* $(0, \infty)$.

Proof. By the theorem of Hille—Yosida, the operator B_γ on $D(B_\gamma; p)$ to $L^p(0, \infty)$ generates a unique contraction semi-group $\{W_\gamma(x); x \geqq 0\}$ of class $(\mathfrak{C}_0)$ in $\mathfrak{E}(L^p(0, \infty))$. The operator $W_\gamma(x)$ $(x > 0)$ is given via the inversion formula (5) through

$$[W_\gamma(x) f]^\wedge(s) = \lim_{\lambda\to\infty} e^{-\lambda x} \sum_{j=0}^\infty \frac{(\lambda^2 x)^j}{j!} [\{R(\lambda; B_\gamma)\}^j f]^\wedge(s) =$$

$$= \lim_{\lambda\to\infty} e^{-\lambda x} \sum_{j=0}^\infty \frac{(\lambda^2 x)^j}{j!} \frac{f^\wedge(s)}{(\lambda + s^\gamma)^j} =$$

$$= \lim_{\lambda\to\infty} \exp\{-\lambda x + \lambda^2 x/(\lambda + s^\gamma)\} f^\wedge(s) = e^{-xs^\gamma} f^\wedge(s),$$

which holds for each fixed s, $\operatorname{Re} s>0$, and all $f\in L^p(0,\infty)$. Since $\exp(-xs^\gamma)$ is the Laplace transform of the density function $p_\gamma(x;\cdot)$ defined in (17) and since $p_\gamma(x;\cdot)$ is non-negative on $(0,\infty)$, belongs to $L^1(0,\infty)$ with $\int_0^\infty p_\gamma(x;u)du=1$ for each $x>0$ (see G. DOETSCH [6, vol. I, p. 263] and K. YOSIDA [13, p. 259 ff]), the representation (16) of $W_\gamma(x)f$ follows. Moreover, the properties of $p_\gamma(x;\cdot)$ assure that $\{W_\gamma(x);\, x\geqq 0\}$ is holomorphic. Thus the function $W_\gamma(x)=w_\gamma(x)f$ on $[0,\infty)$ to $L^p(0,\infty)$ solves the Cauchy problem for each $f\in L^p(0,\infty)$. It remains to prove that the solution is unique. To this end suppose there is a non-trivial null solution $w_{\gamma,0}(x)=w_\gamma(x;\theta)$ on $[0,\infty)$, i.e. $w_\gamma(x;\theta)\neq\theta$ for all $x\geqq 0$. Then for each fixed s with $\operatorname{Re} s>0$,

$$\frac{d}{dx}[w_{\gamma,0}(x)]^\wedge(s)=[B_\gamma w_{\gamma,0}(x)]^\wedge(s)=-s^\gamma[w_{\gamma,0}(x)]^\wedge(s)$$

with

$$\lim_{x\to 0+}[w_{\gamma,0}(x)]^\wedge(s)=0.$$

The solution of this ordinary differential equation is given by $c(s)\exp(-xs^\gamma)$. But by the latter limit condition we obtain that $c(s)=0$ for each s, $\operatorname{Re} s>0$, and consequently $[w_{\gamma,0}(x)]^\wedge(s)=0$ $(\operatorname{Re} s>0)$ or $w_{\gamma,0}(x)=\theta$ for all $x\geqq 0$. This is a contradiction, proving the theorem.

Here we remark that for $\gamma=1$ the solution of II is given by $W_1(x;f_0)=w(x)f_0$ for any $f_0\in D(J^{-1};p)$, where $\{W(x);\, x\geqq 0\}$ is the semi-group of right translations (1) on $L^p(0,\infty)$. However, for $\gamma>1$, Laplace transform methods may not be applied to solve the related Cauchy problem, since $\exp(-s^\gamma)$ $(\operatorname{Re} s>0)$ is not the Laplace integral of a Lebesgue integrable function (see e.g. G. DOETSCH [6, vol. I, p. 163]).

As the solution (16) of the Cauchy problem II is given by a holomorphic semi-group, it is evidently also a solution of the Cauchy problem I of second order for each function f in $L^p(0,\infty)$, where the operator J^{-2} in I is now replaced by $(-B_\gamma)^2=J^{-2\gamma}$ $(0<\gamma<1)$ with domain $D(J^{-2\gamma};p)$ in $L^p(0,\infty)$. Moreover, condition (iii) in I guarantees the uniqueness of the solution (16).

The Cauchy problem I taken in the generalized sense with J^{-2} replaced by $J^{-2\gamma}$ and $\gamma=1/2$ is known to be the formal version of the following boundary-value problem of the heat-conduction equation for a semi-infinite rod $(x\geqq 0)$:

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial w}{\partial t}\qquad (x,t>0)$$

with initial condition $w(x,0)=0$ $(x>0)$ and boundary conditions

$$w(0,t)=f(t)\quad\text{and}\quad\lim_{x\to\infty}w(x,t)=0\qquad (t>0).$$

Also, for $\gamma=1/2$ the density function is explicitly known, thus

$$p_{1/2}(x;t)=\frac{x}{\sqrt{4\pi}}\frac{\exp(-x^2/4t)}{t^{3/2}}\qquad (x,t>0), \tag{18}$$

and the solution $w_{1/2}(x;f)=W_{1/2}(x)f$ takes on the form

$$W_{1/2}(x)f](t)=\frac{x}{\sqrt{4\pi}}\int_0^t f(t-u)\frac{\exp(-x^2/4u)}{u^{3/2}}du\qquad (f\in L^p(0,\infty)), \tag{19}$$

the semi-group property of $W_{1/2}(x)$ being reflected in the functional equation satisfied by the kernel (18):

$$\frac{x_1+x_2}{\sqrt{4\pi}}\frac{\exp(-(x_1+x_2)^2/4t)}{t^{3/2}}=\frac{x_1x_2}{4\pi}\int_0^t\frac{\exp(-x_1^2/4(t-u))\exp(-x_2^2/4u)}{(t-u)^{3/2}u^{3/2}}du$$

$$(x_1,x_2>0,\quad t>0).$$

The latter relation was already noted in 1902 by E. Cesàro, as D. Doetsch [6, vol. III, p. 81, p. 267] remarks. Moreover, the resolvent operator of B is given by

$$[R(\lambda;B_{1/2})f](t)=\int_0^t f(t-u)\left\{\frac{1}{\sqrt{\pi u}}-\lambda e^{\lambda^2u}\operatorname{Erfc}(\lambda\sqrt{u})\right\}du \tag{20}$$

$(\lambda>0; f\in L^p(0,\infty))$, where $\operatorname{Erfc} u=(2/\sqrt{\pi})\int_u^\infty\exp(-v^2)dv$ is the complementary error function.

3. Characterizations of the operator $J^{-\gamma}$, $\gamma>0$

In the foregoing section we have seen that the characterizations (9) and (10) of the domain $D(J^{-\gamma};p)$ of $J^{-\gamma}$ in $L^p(0,\infty)$ and of $J^{-\gamma}$ itself ($\gamma>0$) through the Laplace transform are important auxiliary means for the solution of the Cauchy problem II. However, it is more satisfactory to obtain direct characterizations upon a function $f\in L^p(0,\infty)$ to belong to $D(J^{-\gamma};p)$. This is the object of this section, generalizing at the same time results obtained in [4]. The following lemma gives an evaluation of the Laplace transform of the integral (2).

Lemma 4. *Let $0<\gamma<n$ $(n=1,2,\cdots)$, and let $f\in L(0,b)$ for every $b>0$, $\mathfrak{L}[f](s)$ being absolutely convergent for each s, $\operatorname{Re}s>0$. The Laplace transform of (2) is then given by*

$$\mathfrak{L}\left[\int_\varepsilon^\infty\frac{\Delta_u^nf}{u^{1+\gamma}}du\right](s)=s^\gamma f^\wedge(s)q^\wedge_{\gamma,n}(s\varepsilon)\qquad(\varepsilon>0;\operatorname{Re}s>0), \tag{21}$$

where

(22) $$q^{\wedge}_{\gamma,n}(s)=\frac{1}{s^{\gamma}}\int_{1}^{\infty}\frac{(1-e^{-su})^{n}}{u^{1+\gamma}}\,du \qquad (\operatorname{Re}s>0)$$

is the Laplace transform of the function

(23) $$q_{\gamma,n}(t)=\begin{cases}\dfrac{t^{-1}}{\Gamma(1+\gamma)}\displaystyle\sum_{j=0}^{l}(-1)^{j}\binom{n}{j}(t-j)^{\gamma} & (l<t\leqq l+1;\ l=0,1,\cdots,n-1),\\[2ex] \dfrac{t^{-1}}{\Gamma(1+\gamma)}\displaystyle\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}(t-j)^{\gamma} & (t>n)\end{cases}$$

belonging to $L^{1}(0,\infty)$. *Moreover*

(24) $$C_{\gamma,n}\equiv\lim_{\varepsilon\to 0+}q^{\wedge}_{\gamma,n}(s\varepsilon)=\int_{0}^{\infty}q_{\gamma,n}(u)\,du=\int_{0}^{\infty}\frac{(1-e^{-u})^{n}}{u^{1+\gamma}}\,du$$

and

(25) $$C_{\gamma,n}=\begin{cases}\Gamma(-\gamma)\displaystyle\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}j^{\gamma} & (0<\gamma<n,\ \gamma\ \textit{non-integral})\\[2ex] \dfrac{(-1)^{\gamma+1}}{\gamma!}\displaystyle\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}j^{\gamma}\log j & (\gamma=1,2,\cdots,n-1).\end{cases}$$

Proof. By FUBINI'S theorem we obtain for each fixed s, $\operatorname{Re}s>0$,

$$\int_{0}^{\infty}e^{-st}\,dt\int_{\varepsilon}^{\infty}u^{-1-\gamma}[\Delta^{n}_{u}f](t)\,du=\int_{\varepsilon}^{\infty}u^{-1-\gamma}\,du\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\int_{ju}^{\infty}e^{-st}f(t-ju)\,dt=$$

$$=f^{\wedge}(s)\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\int_{\varepsilon}^{\infty}e^{-sju}u^{-1-\gamma}\,du=s^{\gamma}f^{\wedge}(s)\Big\{\gamma^{-1}(s\varepsilon)^{-\gamma}+$$

$$+\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}j^{\gamma}(s\varepsilon)^{-\gamma}\int_{j}^{\infty}e^{-s\varepsilon u}u^{-1-\gamma}\,du=s^{\gamma}f^{\wedge}(s)q^{\wedge}_{\gamma,n}(s\varepsilon),$$

giving at the same time the representation (22). By the fact that $s^{-\gamma}=\mathfrak{L}[u^{\gamma-1}/\Gamma(\gamma)](s)$ ($\gamma>0$ and $\operatorname{Re}s>0$) as well as by the convolution theorem, this leads to

$$q_{\gamma,n}(u)=\frac{u^{\gamma-1}}{\Gamma(\gamma+1)}+\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}j^{\gamma}h_{j}(u),$$

where

(26) $$h_{j}(u)=\begin{cases}0 & (u<j),\\[1ex] \displaystyle\int_{j}^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}v^{-1-\gamma}\,dv & (u>j).\end{cases}$$

Applying the substitution $(1/v)-(1/u)=t$ in the integral (26), we obtain the representation (23) for $q_{\gamma,n}$.

We now prove that $q_{\gamma,n}$ belongs to $L^1(0,\infty)$. Obviously, $q_{\gamma,n}$ is a continuous function on $(0,\infty)$ and it belongs to $L(0,b)$ for every $b>0$. Moreover, for $\gamma=1,2,\cdots,\cdots,n-1$, $q_{\gamma,n}(t)=0$ for $t>n$. This follows by the fact that the function

$$e^{-\varepsilon t}(1-e^{\varepsilon})^n=\sum_{j=0}^{n}(-1)^j\binom{n}{j}e^{-\varepsilon(t-j)}$$

and its $(n-1)$ derivatives with respect to ε vanish at $\varepsilon=0$. So we may restrict the discussion to non-integral γ, $0<\gamma<n$. By a lengthy calculation one obtains for $t>n$ the representation

$$q_{\gamma,n}(t)=-\frac{\sin\gamma\pi}{t\pi}\int_0^\infty e^{-tv}(1-e^v)^n v^{-\gamma-1}\,dv \qquad (t>n).$$

Hence, for $t>n$ $q_{\gamma,n}(t)$ has a uniquely determined sign: $\operatorname{sgn} q_{\gamma,n}(t)=(-1)^{n-k}$ $(k-1<\gamma<k,\ k=1,2,\cdots,n;\ t>n)$. So it suffices to prove that the limit $\int_a^b q_{\gamma,n}(u)du$ $(b>a>n)$ exists as $b\to\infty$. For $k-1<\gamma<k$, partial integration (k-times) gives

$$\int_a^b u^{\gamma-1}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(1-\frac{j}{u}\right)^{\gamma}du=$$

$$=\sum_{l=0}^{k-1}(-1)^l\frac{u^{\gamma-l}}{\gamma-l}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^l\left(1-\frac{j}{u}\right)^{\gamma-l}\Bigg|_{u=a}^{b}+$$

$$+\sum_{j=0}^{n}(-1)^j\binom{n}{j}(-1)^k j^k\int_a^b u^{\gamma-k-1}\left(1-\frac{j}{u}\right)^{\gamma-k}du.$$

The first sum on the right-hand side of this equation for $u=b$ tends to zero as $b\to\infty$, while the absolute value of the second sum is majorized by

$$\sum_{j=0}^{n}\binom{n}{j}j^k(a-j)^{\gamma-k}/(k-\gamma).$$

Clearly, since $q_{\gamma,n}\in L^1(0,\infty)$ for each fixed s ($\operatorname{Re}s>0$)

$$C_{\gamma,n}\equiv\lim_{\varepsilon\to0+}q^{\wedge}_{\gamma,n}(s\varepsilon)=\int_0^\infty q_{\gamma,n}(u)du,$$

which by the representation (22) with $s=1$ leads to the equation (24). From (24) one may determine the explicit form (25) of $C_{\gamma,n}$.

As a consequence of Lemma 4 we have

Theorem 5. *Let f and g belong to $L^p(0, \infty)$ $(1 \leqq p < \infty)$. Then the following are equivalent for $0 < \gamma < n$:*

(i) $f \in D(J^{-\gamma}; p)$ *with* $J^{-\gamma} f = g$; *i.e.* $s^\gamma f^\wedge(s) = g^\wedge(s)$ $\quad (\operatorname{Re} s > 0)$;

(ii) $\displaystyle \lim_{\varepsilon \to 0+} \left\| \frac{1}{C_{\gamma, n}} \int_\varepsilon^\infty \frac{\Delta_u^n f}{u^{1+\gamma}} du - g \right\|_p = 0.$

Proof. If (i) holds, then by Lemma 4 (21)

$$\frac{1}{C_{\gamma, n}} \int_\varepsilon^\infty \frac{[\Delta_u^n f](t)}{u^{1+\gamma}} du = \frac{1}{C_{\gamma, n}} \int_0^t g(t-u) q_{\gamma, n}\left(\frac{u}{\varepsilon}\right) \frac{du}{\varepsilon} \qquad (\varepsilon > 0) \tag{27}$$

for almost all $t > 0$. Since $q_{\gamma, n} \in L^1(0, \infty)$ with

$$\int_0^\infty q_{\gamma, n}(u)\, du = C_{\gamma, n},$$

the right-hand side of (27) converges to g in the L^p-norm as $\varepsilon \to 0+$, giving (ii). On the other hand, for each fixed s, $\operatorname{Re} s > 0$, by (21)

$$\left| \frac{1}{C_{\gamma, n}} s^\gamma f^\wedge(s) q_{\gamma, n}^\wedge(s\varepsilon) - g^\wedge(s) \right| \leqq \frac{1}{(\sigma p')^{1/p'}} \left\| \frac{1}{C_{\gamma, n}} \int_\varepsilon^\infty \frac{\Delta_u^n f}{u^{1+\gamma}} du - g \right\|_p \qquad (\varepsilon > 0),$$

where $\sigma = \operatorname{Re} s$ and $p^{-1} + p'^{-1} = 1$. Letting $\varepsilon \to 0+$, (*i*) follows by (24).

We remark that Theorem 5 generalizes Theorem 2 of [4] to arbitrary $\gamma > 0$.

Proposition 6. *If a function $f \in L^p(0, \infty)$ belongs to $D(J^{-\gamma_0}; p)$ $(\gamma_0 > 0)$ then f belongs to $D(J^{-\gamma}; p)$ for each $0 < \gamma < \gamma_0$.*

Proof. Let n be an integer such that $n > \gamma_0$. By Theorem 5 we have to prove that the limit in the L^p-norm of

$$\int_\varepsilon^\infty u^{-\gamma-1} \Delta_u^n f du$$

exists as $\varepsilon \to 0+$. Using

$$u^{\gamma_0 - \gamma} = \varepsilon^{\gamma_0 - \gamma} + (\gamma_0 - \gamma) \int_\varepsilon^u v^{\gamma_0 - \gamma - 1}\, dv,$$

one obtains by a change of integration

$$\int_{\varepsilon}^{\infty} \frac{\Delta_u^n f}{u^{1+\gamma}}\, du = \int_{\varepsilon}^{\infty} \frac{\Delta_u^n f}{u^{1+\gamma_0}}\, u^{\gamma_0-\gamma}\, du =$$

$$= (\gamma_0-\gamma) \int_{\varepsilon}^{\infty} v^{\gamma_0-\gamma-1}\, dv \int_{v}^{\infty} \frac{\Delta_u^n f}{u^{1+\gamma_0}}\, du + \varepsilon^{\gamma_0-\gamma} \int_{\varepsilon}^{\infty} \frac{\Delta_u^n f}{u^{1+\gamma_0}}\, du \qquad (\varepsilon>0).$$

But

$$\Big\| \int_{v}^{\infty} u^{-\gamma_0-1} \Delta_u^n f\, du \Big\|_p \leqq 2^n \gamma_0^{-1} v^{-\gamma_0} \|f\|_p$$

for all $f \in L^p(0, \infty)$ and $\leqq \|q_{\gamma,n}\|_1 \, \|J^{-\gamma_0} f\|_p$ for all $f \in D(J^{-\gamma_0}; p)$ by (27). The desired result now follows immediately. As a consequence of Theorem 5 and Proposition 6 we have

Theorem 7. *Let* $0<\gamma<n$ $(n=1, 2, \cdots)$, $\gamma=k+\sigma$ $(k=0, 1, \cdots, n-1$ *and* $0<\sigma\leqq 1)$. *An element* $f\in L^p(0,\infty)$ *belongs to* $D(J^{-\gamma}; p)$ *with* $J^{-\gamma}f=g$ *if and only if* $f\in D(J^{-k}; p)$ *and*

$$\lim_{\varepsilon\to 0+} \Big\| \frac{1}{C_{\sigma,1}} \int_{\varepsilon}^{\infty} \frac{\Delta_u f^{(k)}}{u^{1+\sigma}}\, du - g \Big\|_p = 0 \ \ (0<\sigma<1),$$

$$\lim_{\varepsilon\to 0+} \Big\| \frac{1}{C_{1,2}} \int_{\varepsilon}^{\infty} \frac{\Delta_u^2 f^{(k)}}{u^2}\, du - g \Big\|_p = 0 \ \ (\sigma=1).$$

There are further characterizations of $J^{-\gamma}$ $(\gamma>0)$ and its domain in $L^p(0,\infty)$ by the semi-groups of operators defined in (1) and (16), respectively.

Theorem 8. *Let* $0<\gamma\leqq n$ $(n=1, 2, \cdots)$. *An element* $f\in L^p(0, \infty)$ *belongs to* $D(J^{-\gamma}; p)$ *with* $J^{-\gamma}f=g$ *if and only if*

$$\lim_{x\to 0+} \Big\| \frac{[I-W_{\gamma/n}(x)]^n f}{x^n} - g \Big\|_p = 0 \quad (0<\gamma<n), \qquad \lim_{x\to 0+} \Big\| \frac{\Delta_x^n f}{x^n} - g \Big\|_p = 0 \quad (\gamma=n).$$

The theorem can be proved directly via Laplace transform methods, thus by the results of Sec. 2. On the other hand, it is also a consequence of a general theorem on powers of generators of semi-groups of operators on Banach spaces given in [1] (see also [5, Sec. 2. 2]). Let us finally state one further more general result.

Theorem 9. *Let* $0<\gamma<n\,(n=1, 2, \cdots)$. *For an element* $f\in L^p(0, \infty)$ $(1\leqq p<\infty)$ *the following assertions are equivalent:*

(i) *for* $p=1$: *there is a function* μ *of bounded variation on* $[0, \infty)$ *such that*

$$s^\gamma f^\wedge(s) = \mu^\vee(s) \equiv \int_0^\infty e^{-st}d\mu(t) \qquad (\operatorname{Re} s>0),$$

for $1<p<\infty$: *there is a function* $g\in L^p(0, \infty)$ *with* $s^\gamma f^\wedge(s)=g^\wedge(s)$ $(\operatorname{Re} s>0)$;

$$\text{(ii)} \qquad \left\| \frac{1}{C_{\gamma,n}} \int_\varepsilon^\infty \frac{\Delta_u^n f}{u^{1+\gamma}}\,du \right\|_p = O(1) \qquad (\varepsilon\to 0+);$$

$$\text{(iii)} \qquad \|[I-W_{\gamma/n}(x)]^n f\|_p = O(1) \qquad (x\to 0+).$$

This theorem solves the saturation problem connected with the operators $J^{-\gamma}$ $(\gamma>0)$ in the function space $L^p(0, \infty)$ $(1\leqq p<\infty)$ posed in [2] for $0<\gamma<1$. For a proof of Theorem 9 in case $n=1$ see Theorems 5 and 6 in [4]. Using Lemma 4 these methods may then be easily generalized to arbitrary $n=1, 2, \cdots$ (cf. Theorem 5 above).

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AACHEN, GERMANY, THE TECHNICAL UNIVERSITY

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